

Electromagnetic Properties of the Early Universe

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Detailed physical processes of magnetic field generation from density fluctuations in the pre-recombination era are studied. Solving Maxwell equations and the generalized Ohm's law, the evolutions of the net charge density, the electric current and the electromagnetic field are solved. Unlike most of previous works, we treat electrons and photons as separate components under the assumption of tight coupling. We find that generation of the magnetic field due to density fluctuations takes place only from the second order of both perturbation theory and the tight coupling approximation.

I. INTRODUCTION

Magnetic field generation from density fluctuations in the pre-recombination era ($T \gtrsim 0.3$ eV, where T is temperature of the universe) has been investigated intensively by many authors [1, 2, 3, 4, 5, 6, 7, 8, 9]. Density fluctuations were generated quantum mechanically during the inflationary epoch, evolved linearly during the radiation dominated era, and then acted as the seed for the anisotropy of cosmic microwave background and large scale structure of the universe. Three important ingredients beside dark matter and neutrinos are photons, protons and electrons. They basically behaved as a single fluid until the recombination epoch due to strong coupling through Thomson and Coulomb scatterings. However, because photons preferentially interact with electrons rather than protons, there must be tiny but finite deviation of motion between electrons and protons, that is, the net charge density and the electric current. These are what generate magnetic fields before recombination.

This mechanism has attracted considerable attentions because it could give the seed fields for galactic magnetic fields. Galaxies are observationally known to have magnetic fields of order $1 \mu\text{G}$ while the origin has been a great mystery in modern astrophysics. It is usually considered that if a galaxy has tiny “seed” magnetic fields at its early stage, various hydrodynamical and/or plasma instabilities would amplify the seed fields. This is known as the dynamo mechanism. Accordingly, the problem to find out the origin of galactic magnetic fields reduces to that of seed magnetic fields. Although there have been many mechanisms proposed on the origin of seed fields, the scenario considered here, which is magnetic field generation from density fluctuations, has a great advantage compared to other scenarios that it can give a robust evaluation of generated magnetic fields. This is because density fluctuations themselves have already been measured accurately and a theoretical tool to discuss them, cosmological perturbation theory, has been established firmly. Note that second order density perturbations, which are the next order of linear perturbations, are necessary to be considered for generation of magnetic fields. It is rather lengthy and complicated to solve second order density perturbations and this is why reliable quantitative estimation of seed magnetic fields has not been appeared until recent. For more information on seed magnetic fields and other scenarios, see a comprehensive review [10].

So far, the amplitudes of the generated magnetic fields estimated by different authors are roughly consistent on the horizon scale at recombination, although they differ by several orders on smaller scales [1, 2, 3, 4, 5, 6, 7, 8]. Moreover, the previous studies have focused exclusively on the magnetic field but not on other electromagnetic properties such as the electric field, the net charge density and the electric current. In order to understand the physical processes of magnetic field generation, however, it is undoubtedly important to consider all of them consistently. For example, one may (wrongly) conclude that magnetic fields cannot be generated because tiny deviation of motion between electrons and protons, which induce magnetic fields, could not be maintained due to the shorter timescale of Coulomb scattering between protons and electrons than the one of Thomson scattering between photons and charged particles. One will find out why this argument is not true later in this paper.

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The main purpose of this paper is to clarify all physical processes working on generation of magnetic fields via density fluctuations and to make a physical interpretation of the results of our previous papers [5, 6, 8, 9]. We will solve Maxwell equations and the generalized Ohm's law to express electromagnetic quantities in terms of Thomson scattering term, which is an external force from photons and proportional to velocity difference between photons and charged particles.

As we mentioned earlier, photons and charged particles basically behave as a single fluid such that density fluctuations of them evolve together. Because no magnetic field generation takes place in a single fluid limit, we need some deviation from that limit. But how large deviation do we need? To answer this question, it is helpful to consider the tight coupling approximation [11]. This approximation is based on the fact that the interaction timescale, τ_{int} , between two fluids is much less than the dynamical timescale, τ_{dyn} . The coupling between two fluids is tighter for a smaller interaction timescale and in the limit of $\tau_{\text{int}}/\tau_{\text{dyn}} \rightarrow 0$, we have exact tight coupling and no velocity difference between two fluids. Therefore it is useful to expand physical quantities with respect to the tight coupling parameter $\tau_{\text{int}}/\tau_{\text{dyn}}$ in order to estimate deviation from a single fluid.

Let us give a more specific argument. The scattering timescales for Thomson between photons and charged particles, and Coulomb scatterings between protons and electrons are,

$$\tau_{\text{T}} = \frac{m_p}{\sigma_T \rho_\gamma} \approx 2 \times 10^3 \text{ sec} \left(\frac{1+z}{10^5} \right)^{-4}, \quad (1)$$

$$\tau_{\text{C}} = \frac{m_e}{e^2 n_e \eta} \approx 4 \times 10^{-3} \text{ sec} \left(\frac{1+z}{10^5} \right)^{-3/2}, \quad (2)$$

where m_p is the proton mass, σ_T is the Thomson cross section, ρ_γ is the photon energy density, m_e is the electron mass, e is the electric charge, n_e is the electron number density and z is the redshift. Here η is electric resistivity as,

$$\eta \equiv \frac{\pi e^2 \sqrt{m_e}}{T^{3/2}} \ln \Lambda = 10^{-15} \text{ sec} \left(\frac{1+z}{10^5} \right)^{-3/2} \left(\frac{\ln \Lambda}{10} \right), \quad (3)$$

where $\ln \Lambda$ is the Coulomb logarithm.

On the other hand, the dynamical timescale of fluid may be thought as that of acoustic oscillations. For a given Fourier mode whose wave number is k , the timescale of acoustic oscillations can be written as $\tau_{\text{ac}} = 1/kc_s$, where c_s is the sound velocity. In the radiation dominated epoch, $c_s = c/\sqrt{3}$, where c is speed of light which we take unity hereafter. Accordingly the timescale τ_{ac} can be approximately written as k^{-1} . This simply means that a Fourier mode begins to acoustically oscillate roughly when it enters the horizon. This acoustic oscillation remains until the fluctuation of the scale damps away due to the finite mean free path of photons. This phenomenon, known as Silk damping, occurs at the scale of the diffusion length of photons, k_{Silk}^{-1} . Deviation from tight coupling also occurs due to the finite mean free path so that magnetic field generation will be effective at around the Silk scale, rather than the horizon scale. Thus, the dynamical timescale can be thought to be roughly the Silk scale,

$$k_{\text{Silk}}^{-1} \approx \sqrt{\frac{\tau_{\text{cos}}}{\sigma_T n_e}} \approx 8 \times 10^6 \text{ sec} \left(\frac{1+z}{10^5} \right)^{-5/2}, \quad (4)$$

where τ_{cos} is cosmological time scale, which is the inverse of Hubble parameter, H ,

$$\tau_{\text{cos}} = H^{-1} \approx 5 \times 10^9 \text{ sec} \left(\frac{1+z}{10^5} \right)^{-2}. \quad (5)$$

Then we can have a small parameter for each scattering as,

$$\frac{\tau_{\text{T}}}{\tau_{\text{dyn}}} = 2 \times 10^{-4} \left(\frac{1+z}{10^5} \right)^{-3/2}, \quad \frac{\tau_{\text{C}}}{\tau_{\text{dyn}}} = 5 \times 10^{-10} \left(\frac{1+z}{10^5} \right), \quad (6)$$

where we evaluated the dynamical timescale as k_{Silk}^{-1} . The former is the tight coupling parameter for Thomson scattering and determines the magnitude of velocity difference between photons and charged particles. On the other hand, the latter is for Coulomb scattering and related to the magnitude of the net charge density and the electric current. However, as we will see later, deviation of motion between protons and electrons is suppressed more than expected from the tight coupling approximation because protons and electrons are coupled not only

by Coulomb scattering but also by the electric field. Further, it turns out that deviation of motion between protons and electrons is always much smaller than that between photons and charged particles, even though τ_T becomes smaller than τ_C for $T \gtrsim 10$ keV.

Finally let us give two more important timescales. One is the inverse of plasma frequency,

$$\omega_p^{-1} \equiv \sqrt{\frac{m_e}{e^2 n_e}} = 2 \times 10^{-9} \text{ sec} \left(\frac{1+z}{10^5} \right)^{-3/2}, \quad (7)$$

and the other is the magnetic diffusion timescale,

$$\tau_{\text{diff}} = \frac{\tau_{\text{dyn}}^2}{\eta} \approx 7 \times 10^{28} \text{ sec} \left(\frac{1+z}{10^5} \right)^{-7/2}. \quad (8)$$

We see the following hierarchy for various timescales:

$$\tau_T, \tau_C, \eta, \omega_p^{-1} \ll \tau_{\text{dyn}} \ll \tau_{\text{cos}} \ll \tau_{\text{diff}}. \quad (9)$$

This hierarchy tells us that we can neglect the diffusion of the magnetic field due to electric resistivity. Also, because we focus on the dynamics of the scales much less than the horizon scale, general relativistic effects are expected to be unimportant. Therefore Newtonian treatment will be sufficient and we neglect the cosmological expansion as well. This treatment will make our analysis rather qualitative but clear, and is appropriate for our purpose.

This paper is organized as follows. In the next section, we derive the generalized Ohm's law and an equation for the velocity difference between photons and charged particles. Then in section III, combining the Ohm's law with Maxwell equations, we solve electromagnetic quantities under the assumption of the Thomson term regarded as an external force. The evolution of the Thomson term is investigated in section IV using the tight coupling approximation for Thomson scattering, and combining with the result of section III, electromagnetic quantities are expressed by conventional quantities such as the photon density fluctuation. Finally we give discussion and summary in section V.

II. EQUATIONS OF MOTION

We start from Newtonian equations of motion for fluid densities of photons, protons and electrons, neglecting the cosmological expansion and the pressure of charged particles,

$$\frac{4}{3} \rho_\gamma [\partial_t \vec{v}_\gamma + (\vec{v}_\gamma \cdot \nabla) \vec{v}_\gamma] = -\frac{1}{3} [\nabla \rho_\gamma + \nabla \cdot (\rho_\gamma \Pi_\gamma)] + \vec{C}_{\gamma p}^{(T)} + \vec{C}_{\gamma e}^{(T)} - \frac{4\rho_\gamma}{3} \nabla \Phi, \quad (10)$$

$$m_p n_p [\partial_t \vec{v}_p + (\vec{v}_p \cdot \nabla) \vec{v}_p] = e n_p (\vec{E} + \vec{v}_p \times \vec{B}) + \vec{C}_{pe}^{(C)} + \vec{C}_{p\gamma}^{(T)} - m_p n_p \nabla \Phi, \quad (11)$$

$$m_e n_e [\partial_t \vec{v}_e + (\vec{v}_e \cdot \nabla) \vec{v}_e] = -e n_e (\vec{E} + \vec{v}_e \times \vec{B}) + \vec{C}_{ep}^{(C)} + \vec{C}_{e\gamma}^{(T)} - m_e n_e \nabla \Phi, \quad (12)$$

where ∇ is a derivative with respect to spatial coordinate, \vec{v}_α ($\alpha = \gamma, p, e$) are fluid velocities, Π_γ is anisotropic stress tensor of photons, n_p and n_e are proton and electron number densities, respectively, \vec{E} and \vec{B} are electric and magnetic fields, respectively, and Φ is gravitational potential. Here, $\vec{C}_{ij}^{(T)}$ and $\vec{C}_{ij}^{(C)}$ are collision terms for Thomson and Coulomb scatterings between i and j particles, respectively [5, 8], which are written as

$$\vec{C}_{\gamma p}^{(T)} = -\vec{C}_{p\gamma}^{(T)} = -\frac{m_e^2}{m_p^2} \sigma_T n_p \rho_\gamma \left[(\vec{v}_\gamma - \vec{v}_p) - \frac{1}{4} \vec{v}_p \cdot \Pi_\gamma \right], \quad (13)$$

$$\vec{C}_{\gamma e}^{(T)} = -\vec{C}_{e\gamma}^{(T)} = -\sigma_T n_e \rho_\gamma \left[(\vec{v}_\gamma - \vec{v}_e) - \frac{1}{4} \vec{v}_e \cdot \Pi_\gamma \right], \quad (14)$$

$$\vec{C}_{pe}^{(C)} = -\vec{C}_{ep}^{(C)} = -e^2 n_p n_e \eta (\vec{v}_p - \vec{v}_e). \quad (15)$$

We will rewrite these equations of motion in terms of center-of-mass and relative quantities of charged particles, defined as,

$$n_b \equiv \frac{n_p + \beta n_e}{1 + \beta}, \quad \delta n_{pe} \equiv n_p - n_e, \quad (16)$$

$$\vec{v}_b \equiv \frac{n_p \vec{v}_p + \beta n_e \vec{v}_e}{n_p + \beta n_e}, \quad \delta \vec{v}_{pe} \equiv \vec{v}_p - \vec{v}_e, \quad (17)$$

and conversely,

$$n_p = n_b + \frac{\beta}{1+\beta} \delta n_{pe}, \quad n_e = n_b - \frac{1}{1+\beta} \delta n_{pe}, \quad (18)$$

$$\vec{v}_p = \vec{v}_b + \left[\frac{\beta}{1+\beta} - \frac{\beta}{(1+\beta)^2} \frac{\delta n_{pe}}{n_b} \right] \delta \vec{v}_{pe}, \quad \vec{v}_e = \vec{v}_b - \left[\frac{1}{1+\beta} + \frac{\beta}{(1+\beta)^2} \frac{\delta n_{pe}}{n_b} \right] \delta \vec{v}_{pe}, \quad (19)$$

where $\beta \equiv m_e/m_p$. In terms of the new variables, the net electric charge density and the electric current can be written as,

$$\rho = e(n_p - n_e) = e \delta n_{pe}, \quad (20)$$

$$\vec{j} = e(n_p \vec{v}_p - n_e \vec{v}_e) = e \left[n_b \delta \vec{v}_{pe} + \delta n_{pe} \vec{v}_b - \frac{1-\beta}{1+\beta} \delta n_{pe} \delta \vec{v}_{pe} - \frac{\beta}{(1+\beta)^2} \frac{(\delta n_{pe})^2}{n_b} \delta \vec{v}_{pe} \right]. \quad (21)$$

Then, let us rewrite the equations of motion, Eqs. (10) - (12), in terms of the center-of-mass and relative quantities, Eqs. (16) and (17), and $\delta \vec{v}_{\gamma b} \equiv \vec{v}_\gamma - \vec{v}_b$. We will keep only linear terms in δn_{pe} and $\delta \vec{v}_{pe}$, while we will keep all nonlinear terms in $\delta \vec{v}_{\gamma b}$. The neglect of higher order terms in δn_{pe} and $\delta \vec{v}_{pe}$ will be justified later when we solve all equations and find $\delta n_{pe}/n_b$ and $|\delta \vec{v}_{pe}|$ are much smaller than $|\delta \vec{v}_{\gamma b}|$ for temperatures of interest ($m_e \gtrsim T \gtrsim 0.3$ eV). From $m_e n_e \times$ Eq. (11) $-m_p n_p \times$ Eq. (12), we obtain an equation for $\delta \vec{v}_{pe}$,

$$\begin{aligned} & \frac{m_e}{e(1+\beta)} \left[\partial_t \delta \vec{v}_{pe} + (\vec{v}_b \cdot \nabla) \delta \vec{v}_{pe} + (\delta \vec{v}_{pe} \cdot \nabla) \vec{v}_b \right] \\ &= \vec{E} + \vec{v}_b \times \vec{B} - \frac{1-\beta}{1+\beta} \delta \vec{v}_{pe} \times \vec{B} - \left[en_b \eta + \frac{1+\beta^4}{(1+\beta)^2} \frac{\sigma_T \rho_\gamma}{e} \right] \delta \vec{v}_{pe} - \frac{1-\beta^3}{1+\beta} \frac{\sigma_T \rho_\gamma}{e} \left(\delta \vec{v}_{\gamma b} - \frac{1}{4} \vec{v}_b \cdot \Pi_\gamma \right), \end{aligned} \quad (22)$$

which can be regarded as the generalized Ohm's law. Equation of motion for baryon fluid is obtained from $n_e \times$ Eq. (11) $+n_p \times$ Eq. (12):

$$\partial_t \vec{v}_b + (\vec{v}_b \cdot \nabla) \vec{v}_b = \frac{e}{(1+\beta)m_p} \delta \vec{v}_{pe} \times \vec{B} + \frac{1}{1+\beta} \frac{\sigma_T \rho_\gamma}{m_p} \left[(1+\beta^2) \left(\delta \vec{v}_{\gamma b} - \frac{1}{4} \vec{v}_b \cdot \Pi_\gamma \right) + \frac{1-\beta^3}{1+\beta} \delta \vec{v}_{pe} \right] - \nabla \Phi. \quad (23)$$

On the other hand, Eq. (10) can be rewritten as,

$$\partial_t \vec{v}_\gamma + (\vec{v}_\gamma \cdot \nabla) \vec{v}_\gamma = -\frac{1}{4} \frac{\nabla \rho_\gamma + \nabla \cdot (\rho_\gamma \Pi_\gamma)}{\rho_\gamma} - \frac{3}{4} \sigma_T n_b \left[(1+\beta^2) \left(\delta \vec{v}_{\gamma b} - \frac{1}{4} \vec{v}_b \cdot \Pi_\gamma \right) + \frac{1-\beta^3}{1+\beta} \delta \vec{v}_{pe} \right] - \nabla \Phi, \quad (24)$$

and from Eq. (24) - Eq. (23) we obtain,

$$\begin{aligned} & \partial_t \delta \vec{v}_{\gamma b} + (\vec{v}_\gamma \cdot \nabla) \delta \vec{v}_{\gamma b} + (\delta \vec{v}_{\gamma b} \cdot \nabla) \vec{v}_\gamma - (\delta \vec{v}_{\gamma b} \cdot \nabla) \delta \vec{v}_{\gamma b} \\ &= -\frac{1}{4} \frac{\nabla \rho_\gamma + \nabla \cdot (\rho_\gamma \Pi_\gamma)}{\rho_\gamma} - \frac{e}{(1+\beta)m_p} \delta \vec{v}_{pe} \times \vec{B} \\ & \quad - \frac{1+R}{1+\beta} \frac{\sigma_T \rho_\gamma}{m_p} \left[(1+\beta^2) \left(\delta \vec{v}_{\gamma b} - \frac{1}{4} \vec{v}_b \cdot \Pi_\gamma \right) + \frac{1-\beta^3}{1+\beta} \delta \vec{v}_{pe} \right] \end{aligned} \quad (25)$$

where,

$$R \equiv \frac{3(m_p + m_e)n_b}{4\rho_\gamma} \approx 4 \times 10^{-3} \left(\frac{1+z}{10^5} \right)^{-1}. \quad (26)$$

Among these equations, Eqs. (22), (24) and (25) can be chosen as independent equations which describe the motion of the three fluids. In the context of evolution of cosmic microwave background (CMB) anisotropies, only Eqs. (24) and (25) without $\delta \vec{v}_{pe}$ terms have conventionally been considered. This is based on the assumption that protons and electrons are tightly coupled through Coulomb interaction. As we will show later, this assumption is valid in the sense that the $\delta \vec{v}_{pe}$ terms in Eqs. (24) and (25) are negligible compared to other terms. However, it is obviously impossible to argue electromagnetic properties of the early universe in this approach, i.e., the complete tight coupling limit or taking only the zeroth order of the tight coupling parameter.

In the conventional approach, all physical quantities are expanded according to cosmological perturbation theory. At the zeroth order, the universe is homogeneous and isotropic so that all the vector quantities vanish.

Further, because we are neglecting cosmological expansion densities and resistivity are constant both in time and spatial coordinates: $\rho_\gamma = \rho_\gamma^{(0)}$, $n_p = n_e = n_b = n_b^{(0)}$ and $\eta = \eta^{(0)}$. At the first order, deviations from homogeneity and isotropy are taken into account and typical magnitude of the deviations is about 10^{-5} . Density and tensor perturbations have nonzero values which depend on positions as well as time if they were once generated during the inflation era. Vector-type perturbations, namely divergenceless vectors such as the magnetic field and fluid vorticities, are absent even at the first order since solutions of their perturbations are only decaying mode. They can exist only if we consider the second order. Accordingly, we have,

$$\rho_\gamma(t, \vec{x}) = \rho_\gamma^{(0)} + \rho_\gamma^{(1)}(t, \vec{x}) + \rho_\gamma^{(2)}(t, \vec{x}) + \dots, \quad (27)$$

$$n_b(t, \vec{x}) = n_b^{(0)} + n_b^{(1)}(t, \vec{x}) + n_b^{(2)}(t, \vec{x}) + \dots, \quad (28)$$

$$\eta(t, \vec{x}) = \eta^{(0)} + \eta^{(1)}(t, \vec{x}) + \eta^{(2)}(t, \vec{x}) + \dots, \quad (29)$$

$$\vec{B}(t, \vec{x}) = \vec{B}^{(2)}(t, \vec{x}) + \dots, \quad (30)$$

and other quantities start from the first order, although fluid vorticities are absent at the first order, i.e., $\nabla \times \vec{v}_\gamma^{(1)} = 0$. In this article we will consider up to the second order in cosmological perturbation. We see in the above equations of motion that the Lorentz force term, $\vec{v}_b \times \vec{B}$, and the Hall term, $\delta \vec{v}_{pe} \times \vec{B}$, are neglected because they only appear from the third order.

III. SOLVING MAXWELL + OHM

In this section, we solve Maxwell equations and the generalized Ohm's law to obtain electromagnetic quantities, ρ, \vec{j}, \vec{E} and \vec{B} . Maxwell equations and the charge conservation law are written as,

$$\nabla \cdot \vec{E} = e \delta n_{pe}, \quad (31)$$

$$\partial_t \vec{E} = \nabla \times \vec{B} - e(n_b \delta \vec{v}_{pe} + \delta n_{pe} \vec{v}_b), \quad (32)$$

$$\partial_t \vec{B} = -\nabla \times \vec{E}, \quad (33)$$

$$\partial_t \delta n_{pe} + \nabla \cdot (n_b \delta \vec{v}_{pe} + \delta n_{pe} \vec{v}_b) = 0, \quad (34)$$

and the generalized Ohm's law is obtained from Eq. (22) as,

$$\vec{E} = \frac{m_e}{e(1+\beta)} [\partial_t \delta \vec{v}_{pe} + (\vec{v}_b \cdot \nabla) \delta \vec{v}_{pe} + (\delta \vec{v}_{pe} \cdot \nabla) \vec{v}_b] + e n_b \eta_{\text{eff}} \delta \vec{v}_{pe} + \vec{C}. \quad (35)$$

Here η_{eff} is an effective electric resistivity which includes contributions from Thomson scattering,

$$\eta_{\text{eff}} \equiv \eta + \frac{1+\beta^4}{(1+\beta)^2} \frac{\sigma_T \rho_\gamma}{e^2 n_b} = \eta \left[1 + \frac{1+\beta^4}{(1+\beta)^2} \frac{\tau_C}{\beta \tau_T} \right], \quad (36)$$

Note that the second term in Eq. (36) is dominant for $T \gtrsim 100$ eV. On the other hand, \vec{C} is the Thomson scattering term which is regarded as an external force in this section,

$$\vec{C} \equiv \frac{1-\beta^3}{1+\beta} \frac{\sigma_T \rho_\gamma}{e} \left(\delta \vec{v}_{\gamma b} - \frac{1}{4} \vec{v}_b \cdot \Pi_\gamma \right). \quad (37)$$

The purpose of this section is to express ρ, \vec{j}, \vec{E} and \vec{B} in terms of \vec{C} up to the second order in cosmological perturbation. It is convenient to decompose \vec{C} into scalar and vector parts,

$$\vec{C} = \vec{C}_S + \vec{C}_V, \quad (38)$$

where the scalar part \vec{C}_S can be written by a gradient of a function and \vec{C}_V is divergenceless, $\nabla \cdot \vec{C}_V = 0$. As we stated above, \vec{C}_V is absent at the first order in cosmological perturbation.

A. First order

At the first order, the magnetic field is absent and the equations are,

$$\nabla \cdot \vec{E}^{(1)} = e\delta n_{pe}^{(1)}, \quad (39)$$

$$\partial_t \vec{E}^{(1)} = -en_b^{(0)}\delta \vec{v}_{pe}^{(1)}, \quad (40)$$

$$\partial_t \delta n_{pe}^{(1)} + n_b^{(0)}\nabla \cdot \delta \vec{v}_{pe}^{(1)} = 0, \quad (41)$$

$$\vec{E}^{(1)} = \frac{m_e}{e(1+\beta)}\partial_t \delta \vec{v}_{pe}^{(1)} + en_b^{(0)}\eta_{\text{eff}}^{(0)}\delta \vec{v}_{pe}^{(1)} + \vec{C}^{(1)}. \quad (42)$$

Let us compare the first and second terms in r.h.s. of Eq. (42),

$$\frac{|m_e\partial_t \delta \vec{v}_{pe}^{(1)}/e(1+\beta)|}{|en_b^{(0)}\eta_{\text{eff}}^{(0)}\delta \vec{v}_{pe}^{(1)}|} \sim \frac{m_e k}{e^2 n_b^{(0)}\eta_{\text{eff}}^{(0)}} \sim \begin{cases} k\tau_C \sim 5 \times 10^{-10} \left(\frac{k}{k_{\text{Silk}}}\right) \left(\frac{1+z}{10^5}\right) & (1+z \lesssim 10^6) \\ k\beta\tau_T \sim 10^{-10} \left(\frac{k}{k_{\text{Silk}}}\right) \left(\frac{1+z}{10^7}\right)^{-3/2} & (1+z \gtrsim 10^6) \end{cases}, \quad (43)$$

where we evaluated the time derivative by the wavenumber. Thus, we can neglect the time derivative term when we consider cosmological scales at temperatures $T \lesssim m_e$:

$$\vec{E}^{(1)} = en_b^{(0)}\eta_{\text{eff}}^{(0)}\delta \vec{v}_{pe}^{(1)} + \vec{C}^{(1)}. \quad (44)$$

Actually, this is the leading-order tight coupling approximation for Coulomb scattering. Taking the divergence of this equation, we have,

$$\eta_{\text{eff}}^{(0)}\partial_t \delta n_{pe}^{(1)} + \delta n_{pe}^{(1)} = \frac{1}{e}\nabla \cdot \vec{C}^{(1)}. \quad (45)$$

We can neglect the first term of l.h.s. because

$$\frac{|\eta_{\text{eff}}^{(0)}\partial_t \delta n_{pe}^{(1)}|}{|\delta n_{pe}^{(1)}|} \sim k\eta_{\text{eff}}^{(0)} \sim \begin{cases} k\eta \sim 10^{-22} \left(\frac{k}{k_{\text{Silk}}}\right) \left(\frac{1+z}{10^5}\right) & (1+z \lesssim 10^6) \\ k\eta \frac{\tau_C}{\beta\tau_T} \sim 4 \times 10^{-18} \left(\frac{k}{k_{\text{Silk}}}\right) \left(\frac{1+z}{10^7}\right)^{7/2} & (1+z \gtrsim 10^6) \end{cases}, \quad (46)$$

thus we have,

$$\delta n_{pe}^{(1)} \approx \frac{1}{e}\nabla \cdot \vec{C}^{(1)}. \quad (47)$$

Substituting this into Eq. (41), we can solve for the velocity difference,

$$\delta \vec{v}_{pe}^{(1)} = -\frac{1}{en_b}\partial_t \vec{C}^{(1)}. \quad (48)$$

In general, the rotational part of $\delta \vec{v}_{pe}^{(1)}$ cannot be determined from Eq. (41), but it should vanish at the first order. Then, we obtain the electric field from the Ohm's law as,

$$\vec{E}^{(1)} = \vec{C}^{(1)}, \quad (49)$$

where we neglected $\eta_{\text{eff}}^{(0)}\partial_t \vec{C}^{(1)}$ term. Thus, at the first order, we have,

$$\delta n_{pe}^{(1)} = \frac{1}{e}\nabla \cdot \vec{C}^{(1)}, \quad (50)$$

$$\delta \vec{v}_{pe}^{(1)} = -\frac{1}{en_b^{(0)}}\partial_t \vec{C}^{(1)}, \quad (51)$$

$$\vec{E}^{(1)} = \vec{C}^{(1)}, \quad (52)$$

and we see they also satisfy Eq. (40). Correspondingly, the net charge density and the electric current can be expressed as,

$$\rho^{(1)} = \nabla \cdot \vec{C}^{(1)} \quad (53)$$

$$\vec{j}^{(1)} = -\partial_t \vec{C}^{(1)}. \quad (54)$$

Before we proceed to the second order, let us explain the meaning of the approximations in Eqs. (43) and (46). If we take the divergence of the Ohm's law (42), without neglecting the time derivative term, we have,

$$\frac{1}{(1+\beta)\omega_p^2}\partial_t^2\delta n_{pe}^{(1)} + \eta_{\text{eff}}^{(0)}\partial_t\delta n_{pe}^{(1)} + \delta n_{pe}^{(1)} = \frac{1}{e}\nabla\cdot\vec{C}^{(1)}. \quad (55)$$

Eq. (55) describes the dynamics of charge separation, $\delta n_{pe}^{(1)}$, and can be seen as an equation of damped oscillation with an external force. There are two key timescales. One is the timescale of oscillation, ω_p^{-1} , and another is the damping timescale, $1/(\omega_p^2\eta_{\text{eff}}) \sim \tau_C$. Even though the charge separation and its time derivative are zero initially (outside the horizon), the source term induces the oscillation whose timescale is ω_p^{-1} . Then the oscillation is damped within the timescale τ_C , and charge separation relaxes into the equilibrium value which is nonzero due to the presence of the source term, $e\delta n_{pe}^{(1)} = \nabla\cdot\vec{C}^{(1)}$. Eq. (52) tells us that the force from photons balances with electric field in this state. Because we are focusing on the dynamics of cosmological timescale, it is enough to consider the equilibrium state. The first approximation, Eq. (43), corresponds to the neglect of the first term in l.h.s. of Eq. (55), that is, the neglect of plasma oscillation. Likewise, the second approximation, Eq. (46), corresponds to the neglect of the second term in l.h.s. of Eq. (55). Note that this neglect leads to the absence of magnetic diffusion, that is, magnetic diffusion is important only for very small scales. Thus, both approximations are valid in our context. Similar approximations will also be used at the second order below.

B. Second order

At the second order, the equations are,

$$\nabla\cdot\vec{E}^{(2)} = e\delta n_{pe}^{(2)}, \quad (56)$$

$$\partial_t\vec{E}^{(2)} = \nabla\times\vec{B}^{(2)} - e\left(n_b^{(0)}\delta\vec{v}_{pe}^{(2)} + n_b^{(1)}\delta\vec{v}_{pe}^{(1)} + \delta n_{pe}^{(1)}\vec{v}_b^{(1)}\right), \quad (57)$$

$$\partial_t\vec{B}^{(2)} = -\nabla\times\vec{E}^{(2)}, \quad (58)$$

$$\partial_t\delta n_{pe}^{(2)} + \nabla\cdot\left(n_b^{(0)}\delta\vec{v}_{pe}^{(2)} + n_b^{(1)}\delta\vec{v}_{pe}^{(1)} + \delta n_{pe}^{(1)}\vec{v}_b^{(1)}\right) = 0, \quad (59)$$

$$\begin{aligned} \vec{E}^{(2)} = & \frac{m_e}{e(1+\beta)}\left[\partial_t\delta\vec{v}_{pe}^{(2)} + \left(\vec{v}_b^{(1)}\cdot\nabla\right)\delta\vec{v}_{pe}^{(1)} + \left(\delta\vec{v}_{pe}^{(1)}\cdot\nabla\right)\vec{v}_b^{(1)}\right] \\ & + en_b^{(0)}\eta_{\text{eff}}^{(0)}\left(\delta\vec{v}_{pe}^{(2)} + \frac{n_b^{(1)}}{n_b^{(0)}}\delta\vec{v}_{pe}^{(1)} + \frac{\eta_{\text{eff}}^{(1)}}{\eta_{\text{eff}}^{(0)}}\delta\vec{v}_{pe}^{(1)}\right) + \vec{C}^{(2)}. \end{aligned} \quad (60)$$

As in the case of the first order, the terms in the bracket $[\dots]$ of Eq. (60) are suppressed by the factor in Eq. (43) and can be neglected. Further, the last two terms in the parenthesis (\dots) of Eq. (60) can be neglected compared to $\vec{C}^{(2)}$ because,

$$\left|en_b^{(0)}\eta_{\text{eff}}^{(0)}\frac{n_b^{(1)}}{n_b^{(0)}}\delta\vec{v}_{pe}^{(1)}\right| \sim \left|en_b^{(0)}\eta_{\text{eff}}^{(0)}\frac{\eta_{\text{eff}}^{(1)}}{\eta_{\text{eff}}^{(0)}}\delta\vec{v}_{pe}^{(1)}\right| \sim |k\eta_{\text{eff}}^{(0)}\vec{C}^{(2)}| \ll |\vec{C}^{(2)}|, \quad (61)$$

thus we have simplified second-order Ohm's law as,

$$\vec{E}^{(2)} = en_b^{(0)}\eta_{\text{eff}}^{(0)}\delta\vec{v}_{pe}^{(2)} + \vec{C}^{(2)}. \quad (62)$$

As we did at the first order, we take the divergence of this equation,

$$\eta_{\text{eff}}^{(0)}\partial_t\delta n_{pe}^{(2)} + \delta n_{pe}^{(2)} = \frac{1}{e}\nabla\cdot\vec{C}^{(2)}, \quad (63)$$

and neglecting the first term, we obtain,

$$\delta n_{pe}^{(2)} \approx \frac{1}{e}\nabla\cdot\vec{C}^{(2)}. \quad (64)$$

Then we can solve for $\delta\vec{v}_{pe}^{(2)}$ from Eq. (59) as,

$$\delta\vec{v}_{pe}^{(2)} = -\frac{1}{en_b^{(0)}}\left[\partial_t\left(\vec{C}_S^{(2)} - \frac{n_b^{(1)}}{n_b^{(0)}}\vec{C}^{(1)}\right) + (\nabla\cdot\vec{C}^{(1)})\vec{v}_b^{(1)}\right] + \nabla\times\vec{D}^{(2)}, \quad (65)$$

where $\vec{D}^{(2)}$ is an undetermined vector. From Eq. (60), we have,

$$\vec{E}^{(2)} = \vec{C}^{(2)} + en_b^{(0)} \eta^{(0)} \nabla \times \vec{D}^{(2)}. \quad (66)$$

Here we neglected terms of order $k\eta_{\text{eff}}^{(0)}$. The magnetic field can be obtained from Eq. (58),

$$\vec{B}^{(2)} = - \int dt \nabla \times \vec{E}^{(2)} \quad (67)$$

$$= - \int dt \left[\nabla \times \vec{C}^{(2)} + en_b^{(0)} \eta^{(0)} \nabla \times \nabla \times \vec{D}^{(2)} \right], \quad (68)$$

and the vector $\vec{D}^{(2)}$ is determined by Eq. (57):

$$\begin{aligned} & en_b^{(0)} \left[\eta_{\text{eff}}^{(0)} \partial_t + 1 \right] \nabla \times \vec{D}^{(2)} + \int dt en_b^{(0)} \eta_{\text{eff}}^{(0)} \nabla \times \nabla \times \nabla \times \vec{D}^{(2)} \\ &= - \int dt \nabla \times \nabla \times \vec{C}^{(2)} - \partial_t \vec{C}_V^{(2)}. \end{aligned} \quad (69)$$

Evaluating time and spatial derivatives as wavenumber k and time integration as $1/k$, we pick up only dominant terms:

$$\nabla \times \vec{D}^{(2)} = - \frac{1}{en_b^{(0)}} \int dt \nabla \times \nabla \times \vec{C}^{(2)} - \frac{1}{en_b^{(0)}} \partial_t \vec{C}_V^{(2)}. \quad (70)$$

Thus we obtained all the second-order quantities in terms of $\vec{C}^{(2)}$,

$$\delta n_{pe}^{(2)} = \frac{1}{e} \nabla \cdot \vec{C}^{(2)}, \quad (71)$$

$$\delta \vec{v}_{pe}^{(2)} = - \frac{1}{en_b^{(0)}} \left[\partial_t \left(\vec{C}^{(2)} - \frac{n_b^{(1)}}{n_b^{(0)}} \vec{C}^{(1)} \right) + (\nabla \cdot \vec{C}^{(1)}) \vec{v}_b^{(1)} + \int dt \nabla \times \nabla \times \vec{C}^{(2)} \right], \quad (72)$$

$$\vec{E}^{(2)} = \vec{C}^{(2)}, \quad (73)$$

$$\vec{B}^{(2)} = - \int dt \nabla \times \vec{C}^{(2)}, \quad (74)$$

and correspondingly,

$$\rho^{(2)} = \nabla \cdot \vec{C}^{(2)}, \quad (75)$$

$$\vec{j}^{(2)} = -\partial_t \vec{C}^{(2)} - \int dt \nabla \times \nabla \times \vec{C}^{(2)}. \quad (76)$$

Combining the first and second order results, we have,

$$\rho = \nabla \cdot \vec{C}, \quad (77)$$

$$\vec{j} = -\partial_t \vec{C} - \int dt \nabla \times \nabla \times \vec{C}, \quad (78)$$

$$\vec{E} = \vec{C}, \quad (79)$$

$$\vec{B} = - \int dt \nabla \times \vec{C}, \quad (80)$$

up to the second order. This is one of our main results. Here it should be noted that the vector-type perturbation is absent at the first order so that $\nabla \times \vec{C}^{(1)} = 0$.

Now we expressed the electromagnetic quantities in terms of the Thomson term \vec{C} . We see that the electric current has two contributions in Eq. (78) and the first and second terms balance with the displacement current and rotation of the magnetic field in (57), respectively. Since the Thomson term \vec{C} is the velocity difference between photon and baryon fluctuations, it suffers from Silk damping when k_{Silk}^{-1} exceeds k^{-1} , that is, $\vec{C} \rightarrow 0$ for $k^{-1} < k_{\text{Silk}}^{-1}$. The above solution tells us that the electric current and the magnetic field keep non zero values

even for $\vec{C} = 0$ while the net charge density and the electric field vanish. This is because the diffusion timescale of the magnetic field is so much larger than the dynamical timescale that the magnetic field, once generated, would not damp away and be maintained by the residual electric current (the second term in Eq. (78)) even after the source term would disappear.

Before concluding this section let us comment on the earlier work of [7]. The difference in the treatment between theirs and ours is that we have included the displacement current in Eq. (57) while they have omitted it. We found above that the displacement current leads to the second term in (78) and it is what maintains the magnetic fields after their generation. We attribute the difference in the resultant magnetic fields at small scales between [7] and ours to this contribution from the displacement current.

IV. EVOLUTION OF THOMSON TERM

In this section, we follow the evolution of the Thomson term to discuss magnetic field generation in the language of the tight coupling approximation. This approach was first studied in [9] and it was shown that magnetic field generation is absent at the first order in the tight coupling approximation and starts from the second order. We will confirm this fact and, furthermore, solve the equations explicitly to express the electromagnetic quantities in terms of the conventional quantities such as the density fluctuations of photons. For simplicity, we will ignore the anisotropic stress of photons hereafter. This treatment makes our analysis rather qualitative but it would be helpful in order to see the dynamics of the tight coupling approximation clearly and interpret the numerical analysis of our previous papers [6, 8].

When we consider magnetic field generation, a particularly important quantity is the rotation of the Thomson term,

$$\nabla \times \vec{C}^{(2)} = \frac{1 - \beta^3}{1 + \beta} \frac{\sigma_T \rho_\gamma^{(0)}}{e} \left[\frac{\nabla \rho_\gamma^{(1)}}{\rho_\gamma^{(0)}} \times \delta \vec{v}_{\gamma b}^{(1)} + \nabla \times \delta \vec{v}_{\gamma b}^{(2)} \right]. \quad (81)$$

Here we ignored the anisotropic stress of photons. Thus, we need to solve for $\delta \vec{v}_{\gamma b}$ up to the second order in cosmological perturbation. In Eq. (81) the first term of the r.h.s. is the vector product of the gradient of the photon energy density and the velocity difference between photons and baryons. This was called the "slip term" in [6, 8] and evaluated numerically. On the other hand, the second term is the vorticity difference between photons and baryons and has not been evaluated so far. Although it is hard to compute this term in the framework of general relativistic perturbation theory, we can evaluate it in our simplified formalism.

The evolution equation of $\delta \vec{v}_{\gamma b}$, Eq. (25), without the anisotropic stress of photons is,

$$\partial_t \delta \vec{v}_{\gamma b} + (\vec{v}_\gamma \cdot \nabla) \delta \vec{v}_{\gamma b} + (\delta \vec{v}_{\gamma b} \cdot \nabla) \vec{v}_\gamma - (\delta \vec{v}_{\gamma b} \cdot \nabla) \delta \vec{v}_{\gamma b} = -\frac{1}{4} \frac{\nabla \rho_\gamma}{\rho_\gamma} - \frac{1 + R}{1 + \beta} \frac{\sigma_T \rho_\gamma}{m_p} \left[(1 + \beta^2) \delta \vec{v}_{\gamma b} + \frac{1 - \beta^3}{1 + \beta} \delta \vec{v}_{pe} \right]. \quad (82)$$

We can compare the relative importance of the collision terms from the result of the previous section,

$$\frac{|\delta \vec{v}_{pe}|}{|\delta \vec{v}_{\gamma b}|} \sim \frac{\sigma_T \rho_\gamma k}{e^2 n_b} = \frac{k}{\omega_p^2 \tau_T} \sim 5 \times 10^{-28} \left(\frac{k}{k_{\text{Silk}}} \right) \left(\frac{1 + z}{10^5} \right)^{7/2}. \quad (83)$$

Therefore, we can neglect the $\delta \vec{v}_{pe}$ term and we have,

$$\partial_t \delta \vec{v}_{\gamma b} + (\vec{v} \cdot \nabla) \delta \vec{v}_{\gamma b} + (\delta \vec{v}_{\gamma b} \cdot \nabla) \vec{v} - (\delta \vec{v}_{\gamma b} \cdot \nabla) \delta \vec{v}_{\gamma b} = -\frac{1}{4} \frac{\nabla \rho_\gamma}{\rho_\gamma} - \frac{1 + \beta^2}{1 + \beta} (1 + R) \frac{\sigma_T \rho_\gamma}{m_p} \delta \vec{v}_{\gamma b}. \quad (84)$$

We have also confirmed the conventional assumption in the context of evolution of CMB anisotropies that protons and electrons are so tightly coupled that we can treat them as a single fluid as long as we consider the dynamics of photons and baryons.

We will solve Eq. (84) by employing the tight coupling approximation [9, 11]. As we mentioned in the introduction, this approximation makes use of the fact that the scattering timescale, τ_T , is much shorter than the dynamical timescale, k^{-1} , so that the deviation of the motion of the two fluids is very small. The expansion parameter is the ratio of two timescales,

$$k \tau_T = 2 \times 10^{-4} \left(\frac{k}{k_{\text{Silk}}} \right) \left(\frac{1 + z}{10^5} \right)^{-3/2}. \quad (85)$$

Here we define the deviation of photon and baryon distributions from adiabatic distribution by,

$$\rho_\gamma = \bar{\rho}_\gamma(1 + \Delta_\gamma), \quad n_b = \bar{n}_b(1 + \Delta_b), \quad (86)$$

respectively. These two quantities, Δ_γ and Δ_b , are assumed to be small and we expand them by the tight coupling parameter, Eq. (85), as,

$$\Delta_\gamma = \Delta_\gamma^{(I)} + \Delta_\gamma^{(II)} + \dots, \quad \Delta_b = \Delta_b^{(I)} + \Delta_b^{(II)} + \dots. \quad (87)$$

Fluid velocities are also expanded as,

$$\vec{v}_\gamma = \vec{v} + \vec{v}_\gamma^{(I)} + \vec{v}_\gamma^{(II)} + \dots, \quad \vec{v}_b = \vec{v} + \vec{v}_b^{(I)} + \vec{v}_b^{(II)} + \dots, \quad \delta\vec{v}_{\gamma b} = \delta\vec{v}_{\gamma b}^{(I)} + \delta\vec{v}_{\gamma b}^{(II)} + \dots, \quad (88)$$

where \vec{v} is the common velocity of photons and baryons at the zeroth order and $\delta\vec{v}_{\gamma b}^{(i)} = \vec{v}_\gamma^{(i)} - \vec{v}_b^{(i)}$, ($i = I, II, \dots$). It should be noted that this expansion is independent of cosmological perturbation and each term in Eqs. (87) and (88) can be expanded with respect to cosmological perturbation, e.g. $\Delta_\gamma^{(I)} = \Delta_\gamma^{(I,1)} + \Delta_\gamma^{(I,2)} + \dots$.

Our purpose here is to express $\delta\vec{v}_{\gamma b}$ and then the electromagnetic quantities explicitly in terms of the conventional quantities such as $\bar{\rho}_\gamma^{(1)}$.

A. Continuity Equations

To solve the equation of motion (84), we need to consider the continuity equations of photons and baryons,

$$\partial_t \rho_\gamma + (\vec{v}_\gamma \cdot \nabla) \rho_\gamma + \frac{4\rho_\gamma}{3} \nabla \cdot \vec{v}_\gamma = 0, \quad (89)$$

$$\partial_t n_b + (\vec{v}_b \cdot \nabla) n_b + n_b \nabla \cdot \vec{v}_b = 0. \quad (90)$$

At the zeroth order of the tight coupling approximation, these equations reduce to,

$$\partial_t \bar{\rho}_\gamma + (\vec{v} \cdot \nabla) \bar{\rho}_\gamma + \frac{4\bar{\rho}_\gamma}{3} \nabla \cdot \vec{v} = 0, \quad (91)$$

$$\partial_t \bar{n}_b + (\vec{v} \cdot \nabla) \bar{n}_b + \bar{n}_b \nabla \cdot \vec{v} = 0, \quad (92)$$

which are combined to obtain,

$$(\partial_t + \vec{v} \cdot \nabla) \left(\frac{\bar{n}_b}{\bar{\rho}_\gamma^{3/4}} \right) = 0. \quad (93)$$

This shows that fluctuations of photons and baryons behave adiabatically at this order, as we expected.

As we will see later, we need the relation between Δ_γ and Δ_b at the first order both in the tight coupling approximation and cosmological perturbation to discuss magnetic field generation. At this order, the continuity equations for photons and baryons are,

$$\partial_t \Delta_\gamma^{(I,1)} + \frac{4}{3} \nabla \cdot \vec{v}_\gamma^{(I,1)} = 0, \quad \partial_t \Delta_b^{(I,1)} + \nabla \cdot \vec{v}_b^{(I,1)} = 0, \quad (94)$$

respectively, and then we have,

$$\Delta_b^{(I,1)} = \frac{3}{4} \Delta_\gamma^{(I,1)} + \int dt \nabla \cdot \delta\vec{v}_{\gamma b}^{(I,1)}. \quad (95)$$

B. Equation of Motion

Substituting the expansion Eq. (87) into Eq. (84), we have,

$$\partial_t \delta\vec{v}_{\gamma b} + (\vec{v}_\gamma \cdot \nabla) \delta\vec{v}_{\gamma b} + (\delta\vec{v}_{\gamma b} \cdot \nabla) \vec{v}_\gamma - (\delta\vec{v}_{\gamma b} \cdot \nabla) \delta\vec{v}_{\gamma b} = -\frac{1}{4} \left(\frac{\nabla \bar{\rho}_\gamma}{\bar{\rho}_\gamma} + \nabla \Delta_\gamma \right) - \nu \delta\vec{v}_{\gamma b}, \quad (96)$$

where ν is the collision frequency,

$$\nu \equiv \frac{1 + \beta^2}{1 + \beta} (1 + R) \frac{\sigma_T \rho_\gamma}{m_p} = \bar{\nu} (1 + \Delta_\gamma) \left[\frac{1}{1 + \bar{R}} + \frac{\bar{R}}{1 + \bar{R}} \frac{1 + \Delta_b}{1 + \Delta_\gamma} \right], \quad (97)$$

and barred quantities are the zeroth order of the tight coupling approximation:

$$\bar{\nu} \equiv \frac{1 + \beta^2}{1 + \beta} (1 + \bar{R}) \frac{\sigma_T \bar{\rho}_\gamma}{m_p}, \quad \bar{R} \equiv \frac{3(m_p + m_e) \bar{n}_b}{4 \bar{\rho}_\gamma}. \quad (98)$$

At the zeroth order in tight coupling approximation, the equation of motion reduces to

$$0 = -\frac{1}{4} \frac{\nabla \bar{\rho}_\gamma}{\bar{\rho}_\gamma} - \bar{\nu} \delta \vec{v}_{\gamma b}^{(I)}. \quad (99)$$

This equation can be solved up to the second order in cosmological perturbation:

$$\delta \vec{v}_{\gamma b}^{(I,1)} = -\frac{1}{4\bar{\nu}^{(0)}} \frac{\nabla \bar{\rho}_\gamma^{(1)}}{\bar{\rho}_\gamma^{(0)}}, \quad (100)$$

$$\delta \vec{v}_{\gamma b}^{(I,2)} = -\frac{1}{4\bar{\nu}^{(0)}} \left[\frac{\nabla \bar{\rho}_\gamma^{(2)}}{\bar{\rho}_\gamma^{(0)}} - \left(\frac{\bar{\rho}_\gamma^{(1)}}{\bar{\rho}_\gamma^{(0)}} + \frac{\bar{\nu}^{(1)}}{\bar{\nu}^{(0)}} \right) \frac{\nabla \bar{\rho}_\gamma^{(1)}}{\bar{\rho}_\gamma^{(0)}} \right]. \quad (101)$$

We can show that the rotation of the Thomson term (81), which is the source term of the magnetic field, vanishes at this order, denoting the following relation obtained from the adiabaticity condition, Eq. (93),

$$\frac{\bar{\nu}^{(1)}}{\bar{\nu}^{(0)}} = \frac{4 + 3\bar{R}^{(0)}}{4(1 + \bar{R}^{(0)})} \frac{\bar{\rho}_\gamma^{(1)}}{\bar{\rho}_\gamma^{(0)}}. \quad (102)$$

Let us now consider the next order of the tight coupling approximation in order to argue magnetic field generation. The first order equation is,

$$\partial_t \delta \vec{v}_{\gamma b}^{(I)} + (\vec{v} \cdot \nabla) \delta \vec{v}_{\gamma b}^{(I)} + \left(\delta \vec{v}_{\gamma b}^{(I)} \cdot \nabla \right) \vec{v} = -\frac{1}{4} \nabla \Delta_\gamma^{(I)} - \bar{\nu} \delta \vec{v}_{\gamma b}^{(II)} - \nu^{(I)} \delta \vec{v}_{\gamma b}^{(I)}, \quad (103)$$

and this can be solved as,

$$\delta \vec{v}_{\gamma b}^{(II,1)} = -\frac{1}{4\bar{\nu}^{(0)}} \nabla \Delta_\gamma^{(I,1)} + \frac{1}{4(\bar{\nu}^{(0)})^2} \frac{\partial_t \nabla \bar{\rho}_\gamma^{(1)}}{\bar{\rho}_\gamma^{(0)}}, \quad (104)$$

$$\begin{aligned} \delta \vec{v}_{\gamma b}^{(II,2)} = & -\frac{1}{4\bar{\nu}^{(0)}} \left[\nabla \Delta_\gamma^{(I,2)} - \frac{\bar{\nu}^{(1)}}{\bar{\nu}^{(0)}} \Delta_\gamma^{(I,1)} - \frac{\nu^{(I,1)}}{\bar{\nu}^{(0)}} \frac{\nabla \bar{\rho}_\gamma^{(1)}}{\bar{\rho}_\gamma^{(0)}} \right] \\ & + \frac{1}{4(\bar{\nu}^{(0)})^2} \left[\frac{\partial_t \nabla \bar{\rho}_\gamma^{(2)}}{\bar{\rho}_\gamma^{(0)}} - \left(\frac{\bar{\rho}_\gamma^{(1)}}{\bar{\rho}_\gamma^{(0)}} + \frac{2\bar{\nu}^{(1)}}{\bar{\nu}^{(0)}} \right) \frac{\partial_t \nabla \bar{\rho}_\gamma^{(1)}}{\bar{\rho}_\gamma^{(0)}} - \left(\frac{\partial_t \bar{\rho}_\gamma^{(1)}}{\bar{\rho}_\gamma^{(0)}} + \frac{\partial_t \bar{\nu}^{(1)}}{\bar{\nu}^{(0)}} \right) \frac{\nabla \bar{\rho}_\gamma^{(1)}}{\bar{\rho}_\gamma^{(0)}} \right. \\ & \left. + (\vec{v} \cdot \nabla) \frac{\nabla \bar{\rho}_\gamma^{(1)}}{\bar{\rho}_\gamma^{(0)}} + \left(\frac{\nabla \bar{\rho}_\gamma^{(1)}}{\bar{\rho}_\gamma^{(0)}} \cdot \nabla \right) \vec{v} \right], \end{aligned} \quad (105)$$

where we substituted Eqs. (100) and (101). We see that we have non-zero contribution for the slip term and vorticity difference at this order as

$$\begin{aligned} \frac{\nabla \bar{\rho}_\gamma^{(1)}}{\bar{\rho}_\gamma^{(0)}} \times \delta \vec{v}_{\gamma b}^{(1)} &= \frac{\nabla \bar{\rho}_\gamma^{(1)}}{\bar{\rho}_\gamma^{(0)}} \times \delta \vec{v}_{\gamma b}^{(II,1)} + \nabla \Delta_\gamma^{(I,1)} \times \delta \vec{v}_{\gamma b}^{(I,1)} \\ &= \frac{1}{4(\bar{\nu}^{(0)})^2} \frac{\nabla \bar{\rho}_\gamma^{(1)}}{\bar{\rho}_\gamma^{(0)}} \times \frac{\partial_t \nabla \bar{\rho}_\gamma^{(1)}}{\bar{\rho}_\gamma^{(0)}}, \end{aligned} \quad (106)$$

$$\begin{aligned}
\nabla \times \delta \vec{v}_{\gamma b}^{(2)} &= \frac{1}{4\bar{\nu}^{(0)}} \left[\frac{\nabla \bar{\nu}^{(1)}}{\bar{\nu}^{(0)}} \times \Delta_{\gamma}^{(I,1)} + \frac{\nabla \nu^{(I,1)}}{\bar{\nu}^{(0)}} \times \frac{\nabla \bar{\rho}_{\gamma}^{(1)}}{\bar{\rho}_{\gamma}^{(0)}} \right] \\
&\quad - \frac{1}{4(\bar{\nu}^{(0)})^2} \left[\nabla \left(\frac{\bar{\rho}_{\gamma}^{(1)}}{\bar{\rho}_{\gamma}^{(0)}} + \frac{2\bar{\nu}^{(1)}}{\bar{\nu}^{(0)}} \right) \times \frac{\partial_t \nabla \bar{\rho}_{\gamma}^{(1)}}{\bar{\rho}_{\gamma}^{(0)}} + \nabla \left(\frac{\partial_t \bar{\rho}_{\gamma}^{(1)}}{\bar{\rho}_{\gamma}^{(0)}} + \frac{\partial_t \bar{\nu}^{(1)}}{\bar{\nu}^{(0)}} \right) \times \frac{\nabla \bar{\rho}_{\gamma}^{(1)}}{\bar{\rho}_{\gamma}^{(0)}} \right] \\
&= -\frac{1}{16(\bar{\nu}^{(0)})^2} \frac{4+3\bar{R}^{(0)}}{1+\bar{R}^{(0)}} \frac{\nabla \bar{\rho}_{\gamma}^{(1)}}{\bar{\rho}_{\gamma}^{(0)}} \times \frac{\partial_t \nabla \bar{\rho}_{\gamma}^{(1)}}{\bar{\rho}_{\gamma}^{(0)}} + \frac{1}{16(\bar{\nu}^{(0)})^2} \frac{\bar{R}^{(0)}}{1+\bar{R}^{(0)}} \frac{\nabla \bar{\rho}_{\gamma}^{(1)}}{\bar{\rho}_{\gamma}^{(0)}} \times \int dt \frac{\nabla(\nabla^2 \bar{\rho}_{\gamma}^{(1)})}{\bar{\rho}_{\gamma}^{(0)}}, \quad (107)
\end{aligned}$$

where we used Eq. (102) and the following relation derived from Eq. (95),

$$\frac{\nu^{(I,1)}}{\bar{\nu}^{(0)}} = \frac{4+3\bar{R}^{(0)}}{4(1+\bar{R}^{(0)})} \Delta_{\gamma}^{(I,1)} - \frac{\bar{R}^{(0)}}{4(1+\bar{R}^{(0)})} \frac{1}{\bar{\nu}^{(0)}} \int dt \frac{\nabla^2 \bar{\rho}_{\gamma}^{(1)}}{\bar{\rho}_{\gamma}^{(0)}}. \quad (108)$$

Adding these two contributions, the rotation of the Thomson term can be written as,

$$\nabla \times \vec{C}^{(2)} = \frac{1}{16(\bar{\nu}^{(0)})^2} \frac{1-\beta^3}{1+\beta} \frac{\sigma_T \bar{\rho}_{\gamma}^{(0)}}{e} \frac{\bar{R}^{(0)}}{1+\bar{R}^{(0)}} \frac{\nabla \bar{\rho}_{\gamma}^{(1)}}{\bar{\rho}_{\gamma}^{(0)}} \times \left[\frac{\partial_t \nabla \bar{\rho}_{\gamma}^{(1)}}{\bar{\rho}_{\gamma}^{(0)}} + \int dt \frac{\nabla(\nabla^2 \bar{\rho}_{\gamma}^{(1)})}{\bar{\rho}_{\gamma}^{(0)}} \right]. \quad (109)$$

Below we show the leading order quantity for electromagnetic quantities. All quantities except the magnetic field need just $\delta \vec{v}_{\gamma b}^{(I,1)}$. On the other hand, as we saw above, the magnetic field vanishes at the first order in tight coupling approximation and the second order terms, i.e., $\delta \vec{v}_{\gamma b}^{(II,1)}$ and $\delta \vec{v}_{\gamma b}^{(II,2)}$ are necessary to have a nonzero slip term and vorticity difference in Eq. (81), respectively.

$$\rho^{(1)} = -\frac{1}{4} \frac{1-\beta^3}{1+\beta^2} \frac{1}{1+\bar{R}^{(0)}} \frac{m_p}{e} \frac{\nabla^2 \bar{\rho}_{\gamma}^{(1)}}{\bar{\rho}_{\gamma}^{(0)}}, \quad \delta n_{pe}^{(1)} = -\frac{1}{4} \frac{1-\beta^3}{1+\beta^2} \frac{1}{1+\bar{R}^{(0)}} \frac{m_p}{e^2} \frac{\nabla^2 \bar{\rho}_{\gamma}^{(1)}}{\bar{\rho}_{\gamma}^{(0)}}, \quad (110)$$

$$\vec{j}^{(1)} = \frac{1}{4} \frac{1-\beta^3}{1+\beta^2} \frac{1}{1+\bar{R}^{(0)}} \frac{m_p}{e} \frac{\partial_t \nabla \bar{\rho}_{\gamma}^{(1)}}{\bar{\rho}_{\gamma}^{(0)}}, \quad \delta \vec{v}_{pe}^{(1)} = \frac{1}{4} \frac{1-\beta^3}{1+\beta^2} \frac{1}{1+\bar{R}^{(0)}} \frac{m_p}{e^2 n_b} \frac{\partial_t \nabla \bar{\rho}_{\gamma}^{(1)}}{\bar{\rho}_{\gamma}^{(0)}}, \quad (111)$$

$$\vec{E}^{(1)} = -\frac{1}{4} \frac{1-\beta^3}{1+\beta^2} \frac{1}{1+\bar{R}^{(0)}} \frac{m_p}{e} \frac{\nabla \bar{\rho}_{\gamma}^{(1)}}{\bar{\rho}_{\gamma}^{(0)}}, \quad (112)$$

$$\vec{B}^{(2)} = -\frac{1}{16} \frac{(1+\beta)(1-\beta^3)}{(1+\beta^2)^2} \frac{\bar{R}^{(0)}}{(1+\bar{R}^{(0)})^3} \frac{m_p^2}{e \sigma_T \bar{\rho}_{\gamma}^{(0)}} \int dt \frac{\nabla \bar{\rho}_{\gamma}^{(1)}}{\bar{\rho}_{\gamma}^{(0)}} \times \left[\frac{\partial_t \nabla \bar{\rho}_{\gamma}^{(1)}}{\bar{\rho}_{\gamma}^{(0)}} + \int dt \frac{\nabla(\nabla^2 \bar{\rho}_{\gamma}^{(1)})}{\bar{\rho}_{\gamma}^{(0)}} \right]. \quad (113)$$

We see that all quantities are expressed by some background quantities and a single perturbed quantity, $\bar{\rho}_{\gamma}^{(1)}$. In Eq. (113), the first term in the bracket is contributed from both the slip term and vorticity difference, while the second term is contributed only from the vorticity difference.

V. DISCUSSION AND SUMMARY

In this paper, we made a physical interpretation of the results of previous studies on magnetic field generation from density fluctuations in the pre-recombination era. This was done by solving Maxwell equations, Ohm's law and an equation for velocity difference between photons and baryons. First we expressed electromagnetic quantities in terms of the Thomson term and studied their behavior. We saw that timescales for Coulomb scattering are so short that charge distribution quickly relaxes into its equilibrium state, which is not charge neutrality but charge separation which balances with the external force from photons. It was also shown that magnetic field and electric current do not vanish even after the source term disappear in contrast to electric field and charge density. Then the Thomson term was obtained by the tight coupling approximation up to second order and electromagnetic quantities were expressed by conventional quantities such as the density fluctuations of photons. We found that the second order terms in the tight coupling play essential roles for generation of the magnetic field.

Let us give some order-of-magnitude estimation of various quantities obtained in this paper. The deviation

of motion between photons and baryons is evaluated as,

$$\begin{aligned} \left| \frac{3}{4} \Delta_\gamma^{(I,1)} - \Delta_b^{(I,1)} \right| &\sim \left| \delta \vec{v}_{\gamma b}^{(1)} \right| \sim \frac{1}{4} \frac{m_p}{\sigma_T \rho_\gamma^{(0)}} k \delta_\gamma = \frac{1}{4} k \tau_T \delta_\gamma \\ &\sim 3 \times 10^{-10} \left(\frac{k}{k_{\text{Silk}}} \right) \left(\frac{1+z}{10^5} \right)^{-3/2} \left(\frac{\delta_\gamma}{10^{-5}} \right). \end{aligned} \quad (114)$$

This is exactly what is expected from the tight coupling approximation. On the other hand, the magnitudes of electromagnetic quantities are,

$$\begin{aligned} \left| \frac{\rho^{(1)}}{en_b^{(0)}} \right| &\sim \left| \delta \vec{v}_{pe}^{(1)} \right| \sim \frac{m_p}{4e^2 \bar{n}_b^{(0)}} k^2 \delta_\gamma = \frac{1}{4\beta} \frac{k^2}{\omega_p^2} \delta_\gamma \\ &\sim 3 \times 10^{-34} \left(\frac{k}{k_{\text{Silk}}} \right)^2 \left(\frac{1+z}{10^5} \right)^2 \left(\frac{\delta_\gamma}{10^{-5}} \right). \end{aligned} \quad (115)$$

This is much smaller than expected from the tight coupling approximation for Coulomb scattering. The reason for this can be seen in section III A, where we used two types of approximation. The first is the conventional tight coupling approximation for Coulomb scattering, Eq. (43), and the second is Eq. (46). Thus electromagnetic quantities are suppressed by two factors,

$$\frac{m_e k}{e^2 n_b \eta_{\text{eff}}} \times k \eta_{\text{eff}} = \frac{k^2}{\omega_p^2}, \quad (116)$$

which is the suppression factor seen in Eq. (115). The second suppression factor can be attributed to the existence of coupling due to the electric field. Equations (114) and (115) justify our strategy on the tight coupling approximation that we solved up to second and first order for Thomson and Coulomb scattering, respectively, dropping the nonlinear terms in δn_{pe} and $\delta \vec{v}_{pe}$ and keeping the nonlinear terms in $\delta \vec{v}_{\gamma b}$, Δ_γ and Db up to second order.

It is instructive to know how equations of motion for photons (10), protons (11) and electrons (12) are balanced. each term of the equations at the leading order.

photons:

$$(\text{acceleration}) : (\text{pressure}) : (\text{Thomson}(\text{with protons})) : (\text{Thomson}(\text{with electrons})) \sim 1 : 1 : \beta^2 R : R, \quad (117)$$

protons:

$$(\text{acceleration}) : (\text{electric field}) : (\text{Thomson}) : (\text{Coulomb}) \sim 1 : 1 : \beta^2 : k\eta, \quad (118)$$

electrons:

$$(\text{acceleration}) : (\text{electric field}) : (\text{Thomson}) : (\text{Coulomb}) \sim \beta : 1 : 1 : k\eta. \quad (119)$$

It can be seen that Coulomb scattering is not important for cosmological scales with $k\eta \ll 1$, which corresponds to the fact that magnetic diffusion is absent at these scales.

In this paper, while we ignored the cosmological expansion and the anisotropic stress of photons, which makes this treatment rather not quantitative but qualitative, we could develop a comprehensive treatment of photon, electron and proton fluids, and successfully understand how the deviation between these fluids are determined. We found out deviation between photons and charged particles is much larger than that of protons and electrons. Our analytic treatment is particularly important on scales much smaller than the cosmological horizon where we can safely ignore the cosmic expansion and the numerical analysis is rather difficult to carry out. To make a precise prediction of the spectrum of the magnetic field, however, we need to evaluate the contribution from vorticity difference together with the anisotropic stress of photons, which we will present in future.

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